Stochastic Continuous Time Models

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Why the Stochastic Continuous Time setting?

Being able to create models in the continuous time setting has a few key advantages:

- Continuous time models can be more intuitive
- The continuous time analog of the Bellman equation the Hamilton-Jacobi-Bellman (HJB) has a unique closed form solution
- These models use continuous stochastic processes for the evolution of variables, which will allow us to examine *distributions* of variables

Intuition

Why are continuous time models more intuitive?

- We might believe some variables evolve continuously
  - Stock prices
  - Productivity/technological progress
  - etc.
- We might also believe that a variable has a continuous pdf and has an approximately continuous distribution

The Hamilton-Jacobi-Bellman Equation

A general HJB equation is:

$$\rho V(x) = \max_c \; u(c) + a(x)V'(x) + \frac{1}{2}b(x)^2 V''(x)$$

with

$$dx = a(x)dt + b(x)dW_t$$

- This can be derived from a discrete Bellman equation using Itô calculus
- It has a unique solution to the value function problem
- This unique solution is something we call a viscosity solution
- It also only requires weak boundary conditions

Deriving The HJB I

An intuitive way to find HJB is to start with the discrete time Bellman equation (Dixit, 1993).

$$V(k,t) = \max_c \; u(c)\Delta t + e^{-\rho \Delta t}\mathbb{E}[V(k+\Delta k, t+\Delta t)]$$

Then, using the power series expansion of $e^{-\rho \Delta t}$:

$$\rho \Delta t V(k,t) = \max_c \; u(c)\Delta t + (1-\rho\Delta t)\mathbb{E}[V(k+\Delta k, t+\Delta t) - V(k,t)]$$

Next we have to use stochastic calculus to find the value of this expectation

Deriving The HJB II

Suppose:

$$\Delta k = a(k)\Delta t + b(k)\Delta W t$$

Where ΔW_t is the increment of the Wiener process or $\varepsilon\sqrt{\Delta t}$

Using Itô's lemma:

$$V(k{+}\Delta k, t{+}\Delta t){-}V(k,t) = V_t(k,t)\Delta t{+}V_k(k,t)(\Delta k){+}\frac{1}{2}V_{kk}(k,t)(\Delta k)^2$$

Carrying through the expectation will yield:

$$\mathbb{E}[V(k+\Delta k, t+\Delta t) - V(k,t)] = \\ V_t(k,t)\Delta t + V_k(k,t)a(k)\Delta t + \frac{1}{2}V_{kk}(k,t)b(k)^2\Delta t$$

Deriving The HJB III

Plugging this into our previous equation:

$$\rho \Delta t V(k,t) = \max_c \; u(c)\Delta t$$
$$+ (1-\rho\Delta t)\big(V_t(k,t) + V_k(k,t)a(k) + \frac{1}{2}V_{kk}(k,t)b(k)^2\big)\Delta t$$

Then if we divide by Δt and take the limit as $\Delta t \to 0$ we get the standard HJB

$$\rho V(k) = \max_c \; u(c) + V_t(k,t) + V_k(k,t)a(k) + \frac{1}{2}V_{kk}(k,t)b(k)^2$$

A Special Case with an Analytical Solution I

- Preferences: $u(c) = \log c$
- Technology: $zF(k) = zk$
- Productivity follows a generic diffusion process:

$$dz = \mu(z)dt + \sigma(z)dW_t$$

- Capital evolves according to:

$$dk = (z - \rho - \delta)kdt$$

- Thus our HJB equation is:

$$\rho V(k,z) = \max_c \; \log c + V_k(k,z)(zk - \delta k - c) + V_z(k,z)\mu(z) + \frac{1}{2}V_{zz}(k,z)\sigma^2(z)$$

A Special Case with an Analytical Solution II

- Now suppose:
  1. $c = \rho k$, thus $dk = (z - \rho - \delta)kdt$
  2. Guess that the value function is of the form:
     - $V(k, z) = \nu(z) + \kappa log(k)$
- Our FOC will be:

$$u'(c) = V_k(k, z) \Rightarrow \frac{1}{c} = \frac{\kappa}{k} \rightarrow c = \frac{k}{\kappa}$$

- plugging this into our HJB equation

$$\rho[\nu(z) + \kappa \log(k)] = \log(k) - \log(\kappa) + \frac{\kappa}{k}[zk - \delta k - k/\kappa] + \nu'(z)\mu(z) + \frac{1}{2}\nu''(z)\sigma^2(z)$$

What is the Viscosity Solution?

- The basic idea is that our value function may have kinks and may not be differentiable
- So, we replace the derivative where it does not exist
- The viscosity solution of an HJB equation will have the following form:

$$\rho v(x^*) \begin{cases} \leq r(x^*, \alpha) + \underset{\alpha \in A}{\phi'(x)} f(x^*, \alpha) & v - \phi \text{ has a local max at } x^* \\ \geq r(x^*, \alpha) + \underset{\alpha \in A}{\phi'(x)} f(x^*, \alpha) & v - \phi \text{ has a local min at } x^* \end{cases}$$

- If v is differentiable at x^* then $v'(x^*) = \phi'(x^*)$

More on the Viscosity Solution

- If there is Brownian motion in our problem we would see "vanishing viscosity"
- i.e. the movements in a viscous fluid would go to zero
- This method helps us find a unique solution because it eliminates solutions with concave kinks
- Our HJB will converge to a unique viscosity solution given three conditions
    1. Monotonicity
    2. Consistency
    3. Stability

Solving a Stochastic Continuous-Time Problem I

Using numerical methods we can solve a standard HJB equation:

$$\rho V(x) = \max_c \; u(c) + \mu(x)V_x + \frac{1}{2}\sigma(x)^2 V_{xx}$$

Where x evolves according to:

$$dx = \mu(x)dt + \sigma(x)dW_t$$

and

$$u(c) = \frac{c^{1-\gamma}}{1-\gamma}$$

Solving a Stochastic Continuous-Time Problem II

Kolmogorov Forward (Fokker-Planck) Equation

- If we want information about the distribution of a parameter we also need to solve the Kolmogorov Forward Equation (KF)
- Suppose we have a diffusion process

$$dx = \mu(x)dt + \sigma(x)dW_t \text{ and } x(0) = x_0$$

- Given an initial distribution $g(x, 0) = g_0(x)$ then $g(x, t)$ satisfies

$$\frac{\partial g(x,t)}{\partial t} = -\frac{\partial}{\partial x}[\mu(x)g(x,t)] + \frac{1}{2}\frac{\partial^2}{\partial x^2}[\sigma^2(x)g(x,t)]$$

A Steady State Solution I

Key Assumptions:

- We are at steady state, i.e. $V(x,t) = V(x,\infty)$
- And $0 = -\frac{\partial}{\partial x}[\mu(x)g(x)] + \frac{1}{2}\frac{\partial^2}{\partial x^2}[\sigma^2(x)g(x)]$
- We can discretize the HJB over our state spaces
- We can then write our partial derivatives as backward or forward differences
- We'll choose the backward or forward difference based on the drift of our state variable

A Steady State Solution II

- First we need to discretize our HJB equation
- We do this by approximating the derivatives of our Value function

$$V_x(x_i) \approx \frac{V_{i+1} - V_i}{\Delta x} \quad \text{or} \quad \frac{V_i - V_{i-1}}{\Delta x}$$

$$V_{xx}(x_i) \approx \frac{V_{i+1} - 2V_i + V_{i-1}}{(\Delta x)^2}$$

A Steady State Solution III

- Thus, the discretized HJB will be:

$$\rho V(x_i) = u(c_i) + V_x(x_i)\mu(x) + \frac{1}{2}V_{xx}(x_i)\sigma(x)^2$$

- Where

$$c_i = (u')^{-1}[V_x(x_i)]$$

- Now that the HJB is discretized we use finite difference method to find the steady state solution

A Steady State Solution IV

The HJB Algorithm, the implicit method:

1. Compute V_x for all x
2. Compute the value of consumption from $c_i = (u')^{-1}[V_x(x_i)]$
3. Implement an upwind scheme to find "correct" V_x
4. Using the coefficients determined by the upwind scheme create a transition matrix for this system
5. Solve the following system of non-linear equations

$$\rho V^{n+1} + \frac{V^{n+1} - V^n}{\Delta} = u(V) + A^n V^{n+1}$$

6. Iterate until $V^{n+1} - V^n \approx 0$

A Steady State Solution V

The KF Algorithm, the implicit method:

1. Discretize the KF equation.
   - This will give us the eigenvalue problem $A^T g = 0$
2. Solve this system for $\tilde{g}$
3. Normalize $\tilde{g}$ to get g

A Time Dependent Solution I

Before you can compute a time dependent system you need:

1. An initial condition for KF
    - This can be found similarly to the steady state value
2. A terminal condition for the HJB

A Time Dependent Solution II

The HJB Algorithm:

1. Compute V_x for all x
2. Compute the value of consumption from $c_i = (u')^{-1}[V_x(x_i)]$
3. Implement an upwind scheme to find "correct" V_x
4. Using the coefficients determined by the upwind scheme create a transition matrix for this system
5. Solve the following system of non-linear equations iterating backward in time from the steady state

$$\rho V^{t+1} + \frac{V^{t+1} - V^t}{\Delta} = u^{t+1} + A^t V^{t+1}$$

A Time Dependent Solution III

The KF Algorithm:

1. Load the transition matrix found by solving the HJB, starting from A_1
   - This will give us the eigenvalue problem

$$g_{t+1} = (I - A_t^T dt)^{-1} g_t$$

   - There is no need for rescaling since this scheme preserves mass
2. Repeat for all time periods

A Time Dependent Solution with Shocks

The Algorithm:

1. Compute the steady state, with idiosyncratic shocks
2. Linearize the system about the steady state
   - This requires automatic differentiation
3. If necessary reduce the model
   - Distribution Reduction
   - Value Function Reduction
4. Solve the linearized (reduced) system
5. Analyze aggregate shocks to this system using the time dependent equations

Skip to end

A Krusell-Smith Model I

From Ahn et al. (2018).

- Agents have preferences described by the following utility function
$$\mathbb{E}_0 = \int_0^\infty e^{-\rho t} \frac{c_{jt}^{1-\theta}}{1-\theta} dt$$
- Also households have idiosyncratic labor productivity $z_{jt} \in \{z_L, z_H\}$.
  - Households switch between these two values according to a Poisson process with frequency λ_L and λ_H

A Krusell-Smith Model II

- A representative firm has the following production function

$$Y_t = e^{Z_t} K_t^{\alpha} N_t^{1-\alpha}$$

- Where Z_t evolves according to the following process

$$dZ_t = -\eta Z_t dt + \sigma dW_t$$

A Krusell-Smith Model III

Equilibrium in this model is given by

$$\rho v_t(a,z) = \max_c \ u(c) + \partial_a v_t(a,z)(w_t z + r_t a - c) + \lambda_z (v_t(a,z') - v_t(a,z)) + \frac{1}{dt}\mathbb{E}_t[dv_t(a,z)] \quad (1)$$

$$\frac{dg_t(a,z)}{dt} = -\partial_a[s_t(a,z)g_t(a,z)] - \lambda_z g_t(a,z) + \lambda_{z'} g_t(a,z') \quad (2)$$

A Krusell-Smith Model IV

And by the following conditions

$$w_t = (1-\alpha)e^{Z_t}K_t^{\alpha}\bar{N}^{-\alpha} \tag{3}$$

$$r_t = \alpha e^{Z_t}K_t^{\alpha-1}\bar{N}^{1-\alpha} - \delta \tag{4}$$

$$K_t = \int a g_t(a,z) da dz \tag{5}$$

With optimal savings policy

$$s_t(a,z) = w_t z + r_t a - c_t(a,z) \tag{6}$$

A Krusell-Smith Model V

The steady state for this system is given by

$$\rho v(a,z) = \max_c \; u(c) + \partial_a v(a,z)(wz + ra - c)\lambda_z(v(a,z') - v(a,z)) \quad (1)$$

$$0 = -\partial_a[s(a,z)g(a,z)] - \lambda_z g(a,z) + \lambda_{z'} g(a,z') \quad (2)$$

$$w = (1-\alpha)K_t^{\alpha}\bar{N}^{-\alpha} \quad (3)$$

$$r = \alpha K_t^{\alpha-1}\bar{N}^{1-\alpha} - \delta \quad (4)$$

$$K = \int ag(a,z)dadz \quad (5)$$

With optimal savings policy

$$s(a,z) = wz + ra - c(a,z) \quad (6)$$

A Krusell-Smith Model VI

The discretized steady state is the solution to:

$$\rho v = u(v) + A(v; p)v \tag{1}$$

$$0 = A(v; p)^T g \tag{2}$$

$$p = F(g) \tag{3}$$

A Krusell-Smith Model VII

After finding the steady-state we linearize the following system:

$$\rho v_t = u(v_t) + A(v_t; p_t) v_t + \frac{1}{dt} \mathbb{E}_t dv_t \tag{1}$$

$$\frac{\partial g_t}{\partial t} = A(v_t; p_t)^T g_t \tag{2}$$

$$dZ_t = -\eta Z_t dt + \sigma dW_t \tag{3}$$

$$p_t = F(g_t; Z_t) \tag{4}$$

A Krusell-Smith Model VIII

The first order Taylor expansion of this system can be written as:

$$\mathbb{E}_t \begin{bmatrix} d\hat{v}_t \\ d\hat{g}_t \\ dZ_t \\ 0 \end{bmatrix} = \begin{bmatrix} B_{gg} & 0 & 0 & B_{vp} \\ B_{gv} & B_{gg} & 0 & B_{gp} \\ 0 & 0 & -\eta & 0 \\ 0 & B_{pg} & B_{pZ} & -I \end{bmatrix} \begin{bmatrix} \hat{v}_t \\ \hat{g} \\ Z_t \\ \hat{p}_t \end{bmatrix} dt$$

A Krusell-Smith Model IX

The solution to this system will be: After finding the steady-state we linearize the following system:

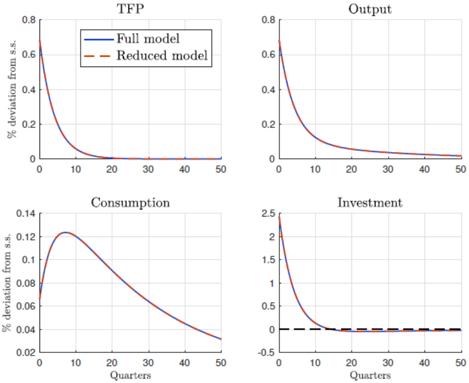
$$\hat{v}_t = D_{vg}\hat{g}_t + D_{vZ}Z_t \tag{1}$$

$$\frac{\partial \hat{g}_t}{\partial t} = (B_{gg} + B_{gp}B_{pg} + B_{gv}D_{vg})\hat{g}_t + (B_{gp}B_{pZ} + B_{gv}D_{vz})Z_t \tag{2}$$

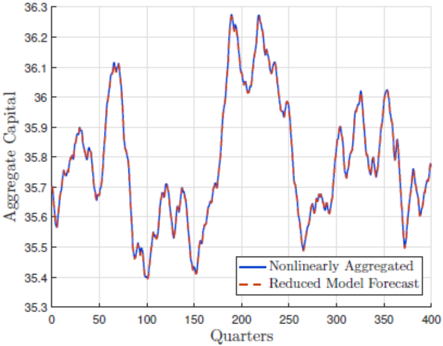
$$dZ_t = -\eta Z_t dt + \sigma dW_t \tag{3}$$

$$\hat{p}_t = B_{pg}\hat{g}_t + B_{pZ}Z_t \tag{4}$$

Results I


TFP
Output
Consumption
Investment
Full model
Reduced model
% deviation from s.s.
% deviation from s.s.
Quarters
Quarters
0.8
0.6
0.4
0.2
0
0.14
0.12
0.1
0.08
0.06
0.04
0.02
2.5
2
1.5
1
0.5
0
-0.5
0
10
20
30
40
50

Results II


36.3
36.2
36.1
36
35.9
35.8
35.7
35.6
35.5
35.4
35.3
Aggregate Capital
0
50
100
150
200
250
300
350
400
Quarters
Nonlinearly Aggregated
Reduced Model Forecast

A Two Asset HANK Model I

Each household has preferences given by

$$\mathbb{E}_0 \int_0^\infty e^{-(\rho+\zeta)t} \log c_{jt} dt \tag{1}$$

They hold liquid or illiquid assets b_t and a_t

$$\frac{db_{jt}}{dt} = (1-\tau)wz_{jt} + T + r^b b_{jt} - \chi(d_{jt}, a_{jt}) - c_{jt} - d_{jt} \tag{2}$$

$$\frac{da_{jt}}{dt} = r_t^a a_{jt} + d_{jt} \tag{3}$$

labor productivity z_{jt} follows a discrete-state Poisson process and switch states with Poisson intensity $\lambda_{zz'}$

A Two Asset HANK Model II

There is a representative firm with the Cobb-Douglas production function

$$Y_t = e^{Z_t} K_t^{\alpha} \bar{L}^{1-\alpha} \tag{4}$$

where

$$dZ_t = -\eta Z_t dt + \sigma dW_t \tag{5}$$

The government adjusts each period to meet the following constraint:

$$\int_0^1 \tau w_t z_{jt} dj = G_t + \int_0^1 T dj \tag{6}$$

The asset market clearing condition is:

$$K_t = \int_0^1 a_{jt} dj \tag{7}$$

A Two Asset HANK Model III

The household's HJB will be:

$$
\begin{aligned}
(\rho + \zeta) v_t(a,b,z) = \max_{c,d} \ & \log c \\
& + \partial_b v_t(a,b,z)((1-\tau) w z_{jt} + T + r^b b_{jt} - \chi(d_{jt}, a_{jt}) - c_{jt} - d_{jt}) \\
& + \partial_a v_t(a,b,z)(r_t^a a_{jt} + d_{jt}) \\
& + \sum_{z'} \lambda_{zz'}(v_t(a,b,z') - v_t(a,b,z)) + \frac{1}{dt}\mathbb{E}_t[dv_t(a,b,z)]
\end{aligned}
$$

A Two Asset HANK Model IV

$$\frac{dg_t(a,b,z)}{dt} = -\partial_a\big(s_t^a(a,b,z)g_t(a,b,z)\big) - \partial_b\big(s_t^b(a,b,z)g_t(a,b,z)\big)$$
$$- \sum_{z'} \lambda_{zz'} g_t(a,b,z) + \sum_{z'} \lambda_{z'z} g_t(a,b,z)$$
$$- \zeta g_t(a,b,z) + \zeta\delta(a)\delta(b)g^*(z)$$

A Two Asset HANK Model V

Equilibrium prices will solve:

$$r_t^a = \alpha e^{Z_t} K_t^{\alpha-1} \bar{L}^{1-\alpha} - \delta \tag{8}$$

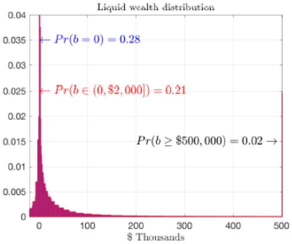
$$w_t = (1-\alpha) e^{Z_t} K_t^{\alpha} \bar{L}^{-\alpha} \tag{9}$$

Market clearing will be determined by:

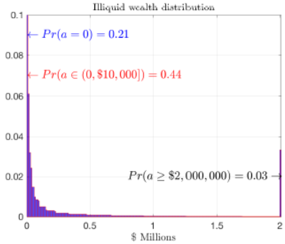
$$K_t = \int a g_t(a, b, z) da db dz \tag{10}$$

$$B = \int b g_t(a, b, z) da db dz \tag{11}$$

A Two Asset HANK Model VI

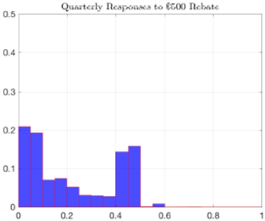


(a) Liquid assets b

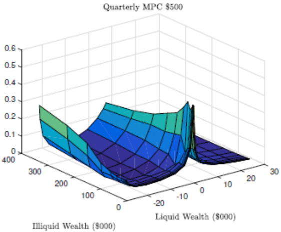


(b) Illiquid assets a

A Two Asset HANK Model VII

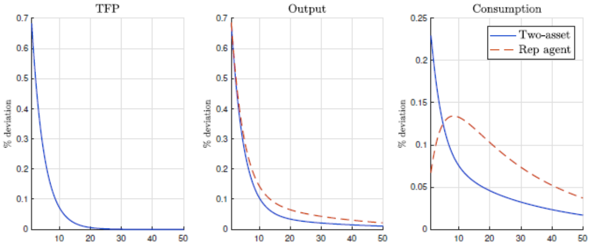


(a) Distribution in Steady State

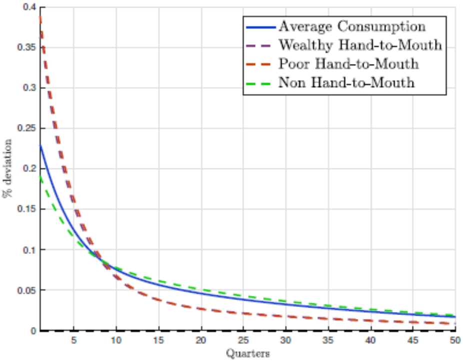


(b) MPC Function

A Two Asset HANK Model VIII



A Two Asset HANK Model IX



Conclusion

- Modeling large complicated markets with heterogeneity is efficient in this setting
  - Krusell-Smith model: 0.116-0.267 sec (2016 Mac-Book Pro)
  - Two Asset HANK: 148.14-286.24 sec (2016 Mac-Book Pro)
- The inequality shown in these models is an important feature not represent in representative agents models
- In this setting we can further explore inequality using distributions
- It would be better to focus on using microdata that captures the distribution of variables in the future

Relevant Literature I

Income and Wealth Distribution in Macroeconomics: A Continuous-Time Approach
by Achdou, Han, Lasry, Lions, & Moll (Forthcoming)

Monetary Policy According to HANK
by Kaplan, Moll, & Violante (2018)

When Inequality Matters for Macro and Macro Matters for Inequality
by Ahn, Kaplan, Moll, Winberry, & Wolf (2018)

Identification and Estimation of Heterogeneous Agent Models: A Likelihood Approach
by Parra-Alvarez, Posch, & Wang (CREATES Working paper)

Lifetime Portfolio Selection Under Uncertainty - Continuous-Time Case
by Merton (1969)

Viscosity Solutions of Hamilton-Jacobi Equations
by Crandall & Lions (1983)

Heterogeneous Households Under Uncertainty
by Pietro Veronesi (NBER Working paper)

Relevant Literature II

Continuous-Time Finance
by Merton (1992)

The Art of Smooth Pasting
by Dixit (1992)

Investment under Uncertainty
by Dixit & Pindyck (1994)

The Economics of Inaction
by Stokey (2009)